Practical Simulation of Surface Tension Flows

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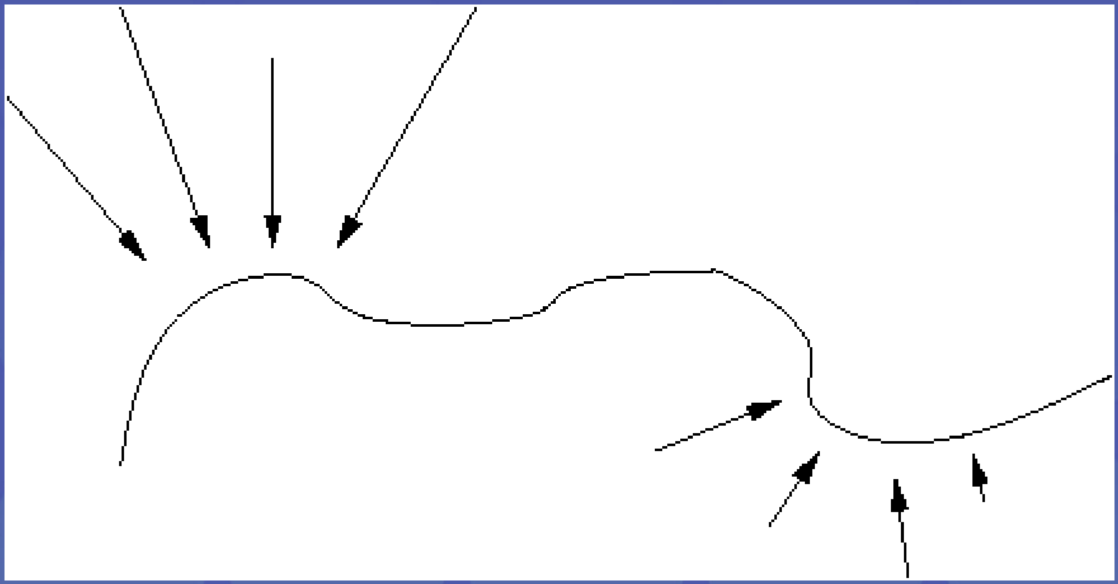
Overview

- Surface Tension
- Capillary Waves - scale issues
- Time Splitting
- Results

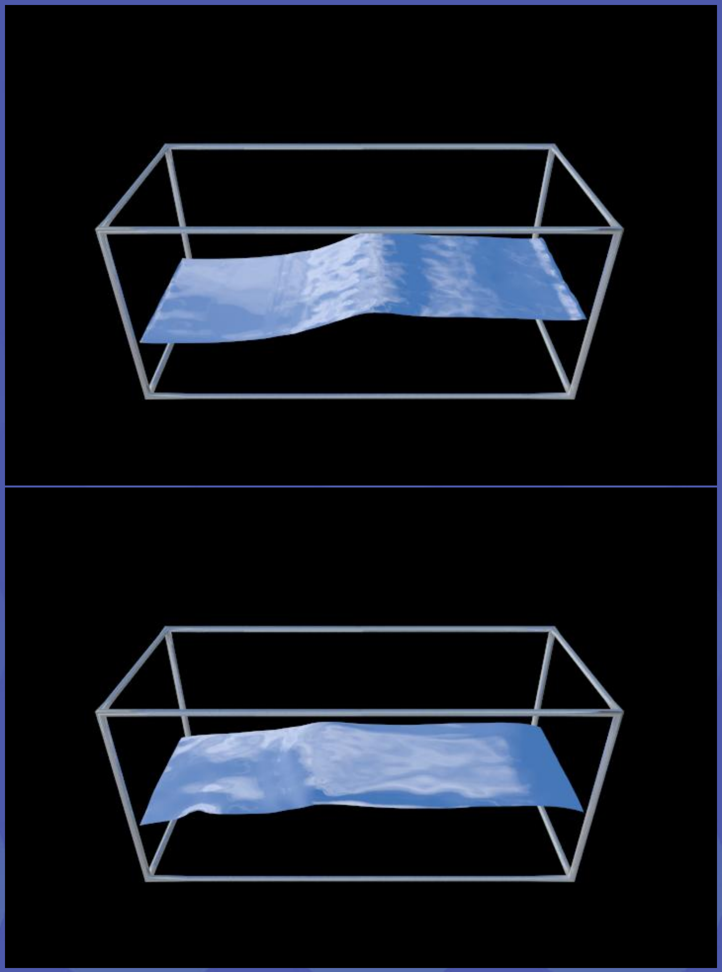
Surface Tension

- Surface Tension Force localized to the interface
- Pushes along negative surface normal
- Causes curvature to be minimized



Surface Tension

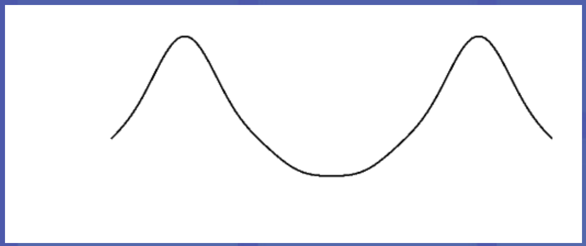
Surface Tension



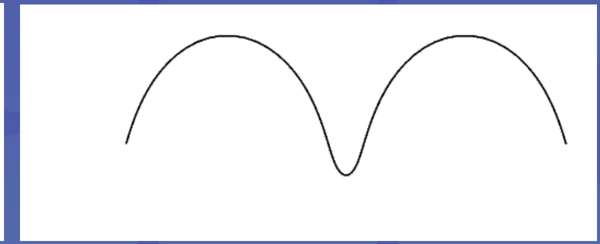
10cm tank. Top: No tension. Bottom: Tension.

Capillary Waves

- Because ST is a restorative force, it causes waves
- Pushes down where curvature is high (crest of wave), water pops up in another place
- Overall effect is wave propagation
- For small waves ($< 2cm$), capillary forces dominate gravity.
- Capillary wave look different from gravity waves



Gravity Wave



Capillary Wave

Capillary Waves

(Image courtesy Fabrice Neyret)

Capillary Waves Speed

- Capillary waves travel *fast*
- Surface tension coefficient σ
- Wavelength ℓ travels with phase speed $\sqrt{2\pi\sigma/\ell}$
- For small surfaces, curvature information travels so fast the surface appears rigid.

Sense of Scale

Cat in the Hat

Numerical Issues

- Wave should not propagate $>$ one grid cell (Δx) per time step (Δt)
- Ignoring this restriction $\Rightarrow$ temporal aliasing of surface
- For smalled wavelength ($2\Delta x$), $\Delta t < \sqrt{\Delta x^3/8\pi\sigma}$
- Time step restriction is $O(\Delta x^{1.5})$.
- CFL condition is $O(\Delta x)$
- Capillary wave restriction is *asymptotically worse*.

Time Splitting

Navier-Stokes Equations:

$$u_t + u \cdot \nabla u + \nabla p = F(S) + g$$

Update Equation:

$$u^{t+\Delta t} = u^t + \Delta t \left[F(S^t) + g - (u^t \cdot \nabla) u^t - \nabla p^t \right]$$
$$S^{t+\Delta t} = advect(S^t, u^t, \Delta t)$$

Pressure term is expensive Must take $\Delta t < \sqrt{\Delta x^3 / 8\pi\sigma}$

But $F(S)$ term changes rapidly, while other terms change slowly...

Time Splitting

Idea: advance different terms at different speeds

- Advance $F(S)$ based on capillary wave rule
- Advance other terms based on CFL
- Couple two equations every so often

Overall, expect to reduce number of times pressure is calculated

Time Splitting

"Split" Update Equation:

$$
\begin{aligned}
u_*^{t+\frac{1}{n}\Delta t} &= u^t + \frac{\Delta t}{n}\left[F(S^t)\right] \\
\cdots \\
u_*^{t+\frac{(n-1)}{n}\Delta t} &= u_*^{t+\frac{(n-2)}{n}\Delta t} + \frac{\Delta t}{n}\left[F(S^{t+\frac{(n-2)}{n}\Delta t}\right] \\
u^{t+\Delta t} &= u^t + \Delta t\left[F(S^{t+\frac{(n-1)}{n}\Delta t}) + g - (u^t \cdot \nabla)u^t - \nabla p^t\right]
\end{aligned}
$$

Results

- Easy to implement
- Speedup improves at higher resolutions (2x is typical)
- Looks pretty good

No ST

Surface Tension

Time Splitting

Thanks

- R&H Simulation Group
- Doug Bloom and Caroline Dahllof
- *Cat in the Hat* production